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A Recurrence Formula for Generalized Divided Differences and Some Applications

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Divided differences are important in connection with interpolation problems. For polynomial interpolation they may be defined by the recurrence formula

$$\begin{aligned}[x_0 \mid f] &= f(x_0)\\ [x_0, ..., x_m \mid f] &= \frac{[x_1, ..., x_m \mid f] - [x_0, ..., x_{m-1} \mid f]}{x_m - x_0}.\end{aligned} \tag{1}$$

We assume that the knots $x_0, ..., x_m$ are all different. An explicit representation is

$$[x_0, ..., x_m \mid f] = \frac{V\begin{pmatrix} p_0, ..., p_{m-1}, f \\ x_0, ..., x_{m-1}, x_m \end{pmatrix}}{V\begin{pmatrix} p_0, ..., p_{m-1}, p_m \\ x_0, ..., x_{m-1}, x_m \end{pmatrix}} \tag{2}$$

where the right-hand side is a quotient of two determinants of the form

$$V\begin{pmatrix} f_0, ..., f_m \\ x_0, ..., x_m \end{pmatrix} := \det f_i(x_k) = \begin{vmatrix} f_0(x_0) & \cdots & f_0(x_m) \\ \vdots & & \vdots \\ f_m(x_0) & \cdots & f_m(x_m) \end{vmatrix}$$

and where

$$p_i(x) := x^i \qquad (i = 0, 1, ...)$$

are the "power-functions."

Basic for these "ordinary" divided differences is the classical complete Čebyšev system $(p_0, ..., p_m)$. We get generalized divided differences of a function f, if we replace this system by an arbitrary Čebyšev-system $(f_0, \ldots, f_m)$ (complete or not),[1] using (2) as definition:[2]

$$\begin{bmatrix} f_0, \ldots, f_m \\ x_0, \ldots, x_m \end{bmatrix} f \Big] := \frac{V\begin{pmatrix} f_0, \ldots, f_{m-1}, f \\ x_0, \ldots, x_{m-1}, x_m \end{pmatrix}}{V\begin{pmatrix} f_0, \ldots, f_{m-1}, f_m \\ x_0, \ldots, x_{m-1}, x_m \end{pmatrix}}. \tag{3}$$

We shall prove, that the divided differences (3) satisfy a recurrence formula in analogy to (1) which allows a simple computation.

THEOREM 1. *Let $I \subseteq \mathbb{R}$ be an interval and $m \geqslant 1$. Let $(f_0, \ldots, f_m)$, $(f_0, \ldots, f_{m-1})$ and (in the case $m \geqslant 2$ also) $(f_0, \ldots, f_{m-2})$ be Čebyšev-systems over I. Consider $m + 1$ different knots $x_i \in I$ $(i = 0, \ldots, m)$. Then*

$$\left[\begin{matrix} f_0, \ldots, f_m \\ x_0, \ldots, x_m \end{matrix} \middle| f\right] = \frac{\left[\begin{matrix} f_0, \ldots, f_{m-1} \\ x_1, \ldots, x_m \end{matrix} \middle| f\right] - \left[\begin{matrix} f_0, \ldots, f_{m-1} \\ x_0, \ldots, x_{m-1} \end{matrix} \middle| f\right]}{\left[\begin{matrix} f_0, \ldots, f_{m-1} \\ x_1, \ldots, x_m \end{matrix} \middle| f_m\right] - \left[\begin{matrix} f_0, \ldots, f_{m-1} \\ x_0, \ldots, x_{m-1} \end{matrix} \middle| f_m\right]}.$$

Proof. Since the case $m = 1$ is trivial we assume $m \geqslant 2$. For abbreviation let

$$N(f) = V\begin{pmatrix} f_0, \ldots, f_{m-2}, f \\ x_1, \ldots, x_{m-1}, x_m \end{pmatrix} V\begin{pmatrix} f_0, \ldots, f_{m-1} \\ x_0, \ldots, x_{m-1} \end{pmatrix} - V\begin{pmatrix} f_0, \ldots, f_{m-2}, f \\ x_0, \ldots, x_{m-2}, x_{m-1} \end{pmatrix} V\begin{pmatrix} f_0, \ldots, f_{m-1} \\ x_1, \ldots, x_m \end{pmatrix}.$$

We show that

$$\left[\begin{matrix} f_0, \ldots, f_m \\ x_0, \ldots, x_m \end{matrix} \middle| f\right] = \frac{N(f)}{N(f_m)}. \tag{4}$$

First note that the denominator of the right-hand side of (4) does not vanish. $N(f_m)$ considered as a function of x_0 ($x_1, \ldots, x_m$ assumed to be fixed) is a linear combination of $f_0, \ldots, f_m$ which has the m zeros $x_1, \ldots, x_m$. It follows from the assumption about $(f_0, \ldots, f_{m-2})$ that the coefficient of f_m does not

[1] [1, p. 1]: The functions $(f_0, \ldots, f_m)$, $f_i \in C[a, b]$, will be called a Čebyšev system over $[a, b]$ when

$$V\begin{pmatrix} f_0, \ldots, f_m \\ x_0, \ldots, x_m \end{pmatrix} > 0$$

for all choices of $x_0 < x_1 < \cdots < x_m$, $x_i \in [a, b]$. The functions $(f_0, \ldots, f_m)$, $f_i \in C[a, b]$, will be referred to as a complete Čebyšev system, if $(f_0, \ldots, f_k)$ is a Čebyšev system for each $k = 0, \ldots, m$.

[2] [2, p. 104].

vanish. Therefore $V(f_m)$ cannot be the zero-function, for otherwise f_m must be a linear combination of $f_0, \ldots, f_{m-1}$, a contradiction. Hence, if the knots are all different, the denominator is different from zero.

$N(f)/N(f_m)$ can be written as a linear combination of $f(x_0), \ldots, f(x_m)$

$$\frac{N(f)}{N(f_m)} = \sum_{k=0}^{m} a_k f(x_k)$$

with real coefficients a_k independent of f. Obviously formula (4) is true for the special functions $f_0, \ldots, f_m$:[3]

$$\sum_{k=0}^{m} a_k f_j(x_k) = \delta_{m,j}, \qquad j = 0, \ldots, m. \tag{5}$$

From this, it follows that (4) is true in general. The real numbers a_k are uniquely determined as solutions of system (5) of linear equations, since its determinant is the generalized van der Monde determinant of the Čebyšev system $(f_0, \ldots, f_m)$. On the other side, the divided difference on the left of (4) is also expressible as a sum of the form

$$\begin{bmatrix} f_0, \ldots, f_m \\ x_0, \ldots, x_m \end{bmatrix} f = \sum_{k=0}^{m} b_k f(x_k)$$

where the coefficients are independent of f and hence solve system (5). Since the solution of (5) is unique, it follows $a_k = b_k$ $(k = 0, \ldots, m)$.

We must divide both nominator and denominator of the right-hand member of (4) by

$$V\begin{pmatrix} f_0, \ldots, f_{m-1} \\ x_1, \ldots, x_m \end{pmatrix} V\begin{pmatrix} f_0, \ldots, f_{m-1} \\ x_0, \ldots, x_{m-1} \end{pmatrix}$$

to obtain Theorem 1.

THEOREM 2. *Let $x_0, \ldots, x_j$, $x_{j+1}, \ldots, x_k$ and $y_0, \ldots, y_j$, $y_{j+1}, \ldots, y_k$ with $x_{j+1} = y_{j+1}, \ldots, x_k = y_k$ be $k + j + 2$ distinct points of an interval I $(0 \leqslant j \leqslant k)$. Suppose $(f_0, \ldots, f_{k+1})$ is a complete Čebyšev system over I and set for $i = 0, \ldots, j$*

$$a_{ik}(f) := \begin{bmatrix} f_0, \ldots, f_k \\ x_0, \ldots, x_i, y_{i+1}, \ldots, y_k \end{bmatrix} f - \begin{bmatrix} f_0, \ldots, f_k \\ x_0, \ldots, x_{i-1}, y_i, \ldots, y_k \end{bmatrix} f.$$

Then we have

$$\begin{bmatrix} f_0, \ldots, f_k \\ x_0, \ldots, x_k \end{bmatrix} f - \begin{bmatrix} f_0, \ldots, f_k \\ y_0, \ldots, y_k \end{bmatrix} f = \sum_{i=0}^{j} a_{ik}(f_{k+1}) \begin{bmatrix} f_0, \ldots, f_{k+1} \\ x_0, \ldots, x_i, y_i, \ldots, y_k \end{bmatrix} f.$$

[3] Kronecker delta.

This generalizes a formula of T. Popoviciu [2, p. 6] for the divided differences (1).

Proof. From (4) we get

$$a_{ik}(f) = a_{ik}(f_{k+1}) \cdot \left[\begin{matrix} f_0, \dots, f_{k+1} \\ x_0, \dots, x_i, y_i, \dots, y_k \end{matrix} \,\middle|\, f\right].$$

Now we sum over $i = 0, \dots, j$. In the sum of the differences $a_{ik}(f)$ all terms cancel with the exception of

$$\left[\begin{matrix} f_0, \dots, f_k \\ x_0, \dots, x_k \end{matrix} \,\middle|\, f\right] - \left[\begin{matrix} f_0, \dots, f_k \\ y_0, \dots, y_k \end{matrix} \,\middle|\, f\right].$$

Theorem 2 states a connection between the divided differences of a function f with respect to the Čebyšev system $(f_0, \dots, f_{k+1})$ and the divided differences of f with respect to the "smaller" system $(f_0, \dots, f_k)$. The following corollary is a direct application of Theorem 2.

COROLLARY. *If the divided differences of a function f with respect to $(f_0, \dots, f_{k+1})$ are bounded on I and the divided differences of f_{k+1} with respect to $(f_0, \dots, f_k)$ too, then the divided differences of f with respect to $(f_0, \dots, f_k)$ are bounded on I.*

Another application of Theorem 1 deals with generalized convex functions. Following S. Karlin and W. Studden, the functions $u_0, \dots, u_m$ will be called an extended complete Čebyšev system, provided $u_i \in C^m[a, b]$, $i = 0, \dots, m$ and

$$V^* \begin{pmatrix} u_0, \dots, u_k \\ x_0, \dots, x_k \end{pmatrix} > 0, \qquad k = 0, \dots, m$$

for all choices $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_m$, $x_i \in [a, b]$. In the case $x_0 = x_1 = \cdots = x_k$, the determinant V^* reduces to the Wronskian determinant $W(u_0, \dots, u_k)$ of the functions $u_0, \dots, u_k$. If $x_{j-1} < x_j = x_{j+1} = \cdots = x_{j+i} < x_{j+i+1}$, we must replace the $i + 1$ columns numbered j through $j + i$ of

$$V \begin{pmatrix} u_0, \dots, u_k \\ x_0, \dots, x_k \end{pmatrix}$$

by the $i + 1$ first columns of the Wronskian $W(u_0, \dots, u_k)$ to obtain the corresponding columns of

$$V^* \begin{pmatrix} u_0, \dots, u_k \\ x_0, \dots, x_k \end{pmatrix}.$$

A function f defined on the interval $[a, b]$ is said to be convex with respect to $(u_0, \ldots, u_k)$ if

$$\begin{bmatrix} u_0, \ldots, u_k \\ x_0, \ldots, x_k \end{bmatrix} f \Big] \geqslant 0$$

for all choices of $x_0 < x_1 < \cdots < x_k$, $x_i \in [a, b]$.[4]

THEOREM 3. *Let f be a differentiable function defined on $[a, b]$ and $(u_0, \ldots, u_m)$ $(m \geqslant 1)$ an extended complete Čebyšev system*

$$u_0(x) = w_0(x)$$

$$u_1(x) = w_0(x) \int_a^x w_1(t_1)\, dt_1$$

$$u_2(x) = w_0(x) \int_a^x w_1(t_1) \int_a^{t_1} w_2(t_2)\, dt_2\, dt_1$$

$$u_m(x) = w_0(x) \int_a^x w_1(t_1) \int_a^{t_1} w_2(t_2) \cdots \int_a^{t_{m-1}} w_m(t_m)\, dt_m \cdots dt_1,$$

where $w_i \in C^{m-i}[a, b]$ are strictly positive functions. Then f is convex with respect to $(u_0, \ldots, u_m)$ if and only if $(f/u_0)'$ is convex with respect to the first "reduced system" $(v_0, \ldots, v_{m-1})$, $v_i = (u_{i+1}/u_0)'$.

This theorem generalizes the well-known fact that a differentiable function f is non-decreasing, convex etc. if and only if the derivative f' is nonnegative, nondecreasing, etc.[5] A proof of this theorem where (in the case $m \geqslant 2$) no use is made of the differentiability of f can be found in [1, p. 393 ff]. But it is rather complicated, for it refers to the fact that a convex function is endowed with substantial continuity and differentiability properties, and as stated by Karlin and Studden [1, p. 381], "the detailed presentation of their proofs is rather elaborate." The following proof of Theorem 3 uses only elementary methods.

To prove the sufficiency of the condition we factor out of

$$V = V\begin{pmatrix} u_0, \ldots, u_{m-1}, f \\ x_0, \ldots, x_{m-1}, x_m \end{pmatrix} \qquad c := \prod_{i=0}^{m} u_0(x_i) > 0$$

[4] Note the little deviation from the definitions 1.1 in [1, p. 375] or 3 in [2, p. 104].

[5] For example, see [3, p. 40].

and subtract from each column its predecessor and expand by minors of the first row

$$V = c \cdot \begin{vmatrix} \frac{u_1}{u_0}(x_1) - \frac{u_1}{u_0}(x_0) & \cdots & \frac{u_1}{u_0}(x_m) - \frac{u_1}{u_0}(x_{m-1}) \\ \vdots & & \\ \frac{f}{u_0}(x_1) - \frac{f}{u_0}(x_0) & \cdots & \frac{f}{u_0}(x_m) - \frac{f}{u_0}(x_{m-1}) \end{vmatrix}.$$

Using the mean value-theorem we obtain

$$V = c \cdot \prod_{i=1}^{m} (x_i - x_{i-1}) \cdot V \binom{v_0, \ldots, v_{m-2}, (f/u_0)'}{z_0, \ldots, z_{m-2}, z_{m-1}},$$

where $x_0 < z_0 < x_1 < z_1 < \cdots < z_{m-1} < x_m$. This proves the sufficiency. To show the necessity we consider the determinant of order m

$$U = U \left[\begin{matrix} x_0, \ldots, x_{m-1} \\ x_0', \ldots, x_{m-1}' \end{matrix} \middle| f \right]$$

$$= \begin{vmatrix} 1 & [x_0, x_0' \mid u_2] & [x_0, x_0' \mid u_3] & \cdots & [x_0, x_0' \mid u_{m-1}] & [x_0, x_0' \mid f] \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & [x_{m-1}, x_{m-1}' \mid u_2] & [x_{m-1}, x_{m-1}' \mid u_3] & \cdots & [x_{m-1}, x_{m-1}' \mid u_{m-1}] & [x_{m-1}, x_{m-1}' \mid f] \end{vmatrix}$$

where the divided differences are taken with respect to (u_0, u_1) and where $x_0 < x_0' < x_1 < x_1' < \cdots < x_{m-1} < x_{m-1}'$. We denote by $y_0, y_1, \ldots, y_{2m-1}$ these points in increasing order and show

$$U = \sum_{i=0}^{m-1} a_i \left[\begin{matrix} u_0, u_1, \ldots, u_m \\ y_i, y_{i+1}, \ldots, y_{i+m} \end{matrix} \middle| f \right] \tag{6}$$

with positive coefficients a_i independent of f. To prove this we subtract from each row of U its predecessor and use the recurrence relation of Theorem 1. Expanding along the first column U reduces to a determinant of order $m - 1$. Its kth row ($k = 1, \ldots, m - 1$) has the form

$$\left[\begin{matrix} u_0, u_1 \\ x_k, x_k' \end{matrix} \middle| h_j \right] - \left[\begin{matrix} u_0, u_1 \\ x_{k-1}, x_{k-1}' \end{matrix} \middle| h_j \right]$$

$$= \left[\begin{matrix} u_0, u_1 \\ x_k, x_k' \end{matrix} \middle| h_j \right] - \left[\begin{matrix} u_0, u_1 \\ x_{k-1}', x_k \end{matrix} \middle| h_j \right] + \left[\begin{matrix} u_0, u_1 \\ x_{k-1}', x_k \end{matrix} \middle| h_j \right] - \left[\begin{matrix} u_0, u_1 \\ x_{k-1}, x_{k-1}' \end{matrix} \middle| h_j \right]$$

$$= \left\{ \left[\begin{matrix} u_0, u_1 \\ x_k, x_k' \end{matrix} \middle| u_2 \right] - \left[\begin{matrix} u_0, u_1 \\ x_{k-1}', x_k \end{matrix} \middle| u_2 \right] \right\} \cdot \left[\begin{matrix} u_0, u_1, u_2 \\ x_{k-1}', x_k, x_k' \end{matrix} \middle| h_j \right]$$

$$+ \left\{ \left[\begin{matrix} u_0, u_1 \\ x_{k-1}', x_k \end{matrix} \middle| u_2 \right] - \left[\begin{matrix} u_0, u_1 \\ x_{k-1}, x_{k-1}' \end{matrix} \middle| u_2 \right] \right\} \cdot \left[\begin{matrix} u_0, u_1, u_2 \\ x_{k-1}, x_{k-1}', x_k \end{matrix} \middle| h_j \right]$$

where $h_j = u_j$ $(j = 2,..., m - 1)$ and $h_m = f$. Assume for the moment that the factors in braces are positive. Since a determinant is a multilinear function of its columns, U can be written as a linear combination of determinants of order $m - 1$ of the same form as U. The coefficients are positive and independent of f, and the elements are now divided differences of order 2. Treating these determinants in the same way, we get after $m - 1$ steps formula (6). At the $(k - 1)$st step there arise coefficients of the form

$$\left\{\begin{bmatrix} u_0, u_1, ..., u_{k-1} \\ t_1, ..., t_k \end{bmatrix} u_k\right] - \left[\begin{matrix} u_0, u_1, ..., u_{k-1} \\ t_0, ..., t_{k-1} \end{matrix} \middle| u_k\right]\right\} \tag{7}$$

with some $t_0 < \cdots < t_k$ from $y_0 < \cdots < y_{2m-1}$; (we show below they are positive). From (6) it follows

$$\lim U\begin{pmatrix} x_0, x_1, ..., x_{m-1} \\ x_0', x_1', ..., x_{m-1}' \end{pmatrix} f\Bigg)$$

$$= \left[\prod_{i=0}^{m-1} v_0(x_i')\right]^{-1} \cdot V\begin{pmatrix} v_0, v_1, ..., v_{m-2}, (f/u_0)' \\ x_0', x_1', ..., x_{m-2}', x_{m-1}' \end{pmatrix} \geqslant 0$$

when $x_i \to x_i'$ $(i = 0,..., m - 1)$. But since v_0 is strictly positive this means that $(f/u_0)'$ is indeed convex over I with respect to $(v_0, ..., v_{m-1})$.

It remains to show the coefficients (7) are positive. The following lemma will be needed.

LEMMA. *Let $1 \leqslant k \leqslant m$ and define $\bar{w}_i = w_i$ $(i = 0,..., k - 1)$ and*

$$\bar{w}_k(t) = \begin{cases} w_k(t), & x \leqslant t \leqslant b \\ 0, & a \leqslant t \leqslant x \end{cases}$$

for some x, $a < x < b$, and

$$\bar{u}(t) = \bar{w}_0(t) \int_a^t \bar{w}_1(t_1) \int_a^{t_1} \bar{w}_2(t_2) \cdots \int_a^{t_{k-1}} \bar{w}_k(t_k)\, dt_k \cdots dt_1 .$$

Then $\bar{u}$ is convex with respect to $(u_0, ..., u_j)$ for $j = 0,..., k$. Moreover

$$\left[\begin{matrix} u_0, ..., u_j \\ t_0, ..., t_j \end{matrix} \middle| \bar{u}\right] \begin{cases} > 0, & x < t_j, \quad a \leqslant t_0 < t_1 < \cdots < t_j \leqslant b \\ = 0, & a \leqslant t_0 < t_1 < \cdots < t_j \leqslant x \end{cases}$$

$(j = 0,..., k)$.

Proof. Proof by induction: In the case $k = 1$ the assertions are trival. Using the following formula for $k > 1$

$$V\binom{u_0, \ldots, u_{k-1}, \bar{u}}{t_0, \ldots, t_{k-1}, t_k}$$
$$= \int_{t_0}^{t_1} \int_{t_1}^{t_2} \cdots \int_{t_{k-1}}^{t_k} \left\{ \prod_{i=0}^{k} u_0(t_i) \right\} \cdot V\binom{v_0, \ldots, v_{k-2}, (\bar{u}/u_0)'}{\eta_0, \ldots, \eta_{k-2}, \eta_{k-1}} d\eta_{k-1} \cdots d\eta_1$$

(see [1, p. 383]) where $v_i = (u_{i+1}/u_0)'$ $(i = 0, \ldots, k-2)$ are the functions of the first "reduced system" of $(u_0, \ldots, u_{k-1})$, the lemma can be reduced to the case $k - 1$.

Now for any $t_0 < t_1 < \cdots < t_k$ the expression in (7) is positive since $N(u_k) > 0$. Indeed, if we choose x in $t_{k-1} < x < t_k$ in the lemma, then

$$\left[\begin{matrix} u_0, \ldots, u_k \\ t_0, \ldots, t_k \end{matrix} \middle| \bar{u}\right] > 0$$

and

$$\left[\begin{matrix} u_0, \ldots, u_{k-1} \\ t_1, \ldots, t_k \end{matrix} \middle| \bar{u}\right] - \left[\begin{matrix} u_0, \ldots, u_{k-1} \\ t_0, \ldots, t_{k-1} \end{matrix} \middle| \bar{u}\right] = \left[\begin{matrix} u_0, \ldots, u_{k-1} \\ t_1, \ldots, t_k \end{matrix} \middle| \bar{u}\right] > 0.$$

Thus $N(\bar{u}) > 0$ and by (4), $N(u_k) > 0$ and (7) holds.

References

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